Borel Circle Squaring

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Dictionary

- A **Borel set** is any set that we can generate from open sets (or equivalently closed) by countable unions and countable intersections.
- A **Borel map** is any map such that preimages of Borel sets are Borel (it suffices to have Borel preimages of open/closed sets).

- Let $G = (V, E)$ be a graph, let $f\colon V \to \mathbb{R}$ be a function and $c\colon E \cup E^* \to \mathbb{R}_0^+$ be a capacity function. Then ϕ is an **f-flow bounded by** c iff for every $(x, y) \in E$ we have $\phi(x, y) = -\phi(y, x)$ and $\phi(x, y) \leq c(x, y)$ and for every $x \in V$ it holds that
$$f(x) = \sum_{y:(x,y)\in E} \phi(x, y)$$
We say that ϕ is an **(ϵ, f)-flow** iff ϕ is symmetric (as above) and for every $x \in V$ it holds that
$$\left| f(x) - \sum_{y:(x,y)\in E} \phi(x, y) \right| < \epsilon$$

- For a finite set $F \subset \mathbb{R}^k$ and $A \subseteq \mathbb{R}^k$ we define **discrepancy** $D(F, A) := \left| \frac{|F\cap A|}{|F|} - \mathcal{L}(A) \right|$.

Group actions. We say that a is an **action** of a group Γ on the space X, we write $a\colon \Gamma \curvearrowright X$, if $a(\gamma, \cdot)$ is a structure-preserving transformation of X and $\gamma \mapsto a(\gamma, \cdot)$ is a group homomorphism.

- a is **free** if it is fixed-point free, i.e. whenever $\gamma \in \Gamma$ and there is $x \in X$ such that $\gamma \cdot_a x = x$, then $\gamma = \mathrm{id}$.
- a is **Borel**, if a is Borel as a map.
- a graph G_a of a is a graph (X, E), where $(x, y) \in E$ iff there is $\gamma \in \Gamma$ with $\|\gamma\|_\infty = 1$ such that $y = \gamma \cdot_a x$.
- for $u \in (\mathbb{T}^k)^d$ we define $a_u\colon \mathbb{Z}^d \curvearrowright \mathbb{T}^k$ as $\sum_{i=1}^k n_i u_i + x$, for any $x \in \mathbb{T}^k$ and $n = (n_1, \ldots, n_d) \in \mathbb{Z}^d$.
- we say that two sets $A, B \subset X$ are **Γ-equidecomposable** iff there is a partition $A_1 \cup \cdots \cup A_n = A$ and elements $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\gamma_1 \cdot_a A_1 \cup \cdots \cup \gamma_n \cdot_a A_n$ is a partition of B.

Theorems

Theorem (1.2)**.** *Suppose $k \geq 1$ and suppose $A, B \subseteq \mathbb{R}^k$ are bounded Borel sets such that $\Delta(A) < k$ and $\Delta(B) < k$ and $\mathcal{L}(A) = \mathcal{L}(B) > 0$. Then A and B are equidecomposable by translations using Borel pieces.*

Lemma (2.1 (Laczkovich))**.** *Suppose $k \geq 1$ and suppose $A \subseteq \mathbb{R}^k$ is a measurable set such that $\Delta(A) < k$ and $\mathcal{L}(A) > 0$. Then for almost every $u \in (\mathbb{T}^k)^d$ there is $\epsilon > 0$ and $M > 0$ such that for every $x \in \mathbb{T}^k$ and $N \in \mathbb{N}$ we have $D(R_N \cdot_{a_u} x, A) \leq \frac{M}{N^{1+\epsilon}}$, where $R_N := \{(n_1, \ldots, n_d) : 0 \leq n_i < N\}$.*

Lemma (5.4)**.** *Suppose $d \geq 2$. Let $a \colon \mathbb{Z}^d \curvearrowright X$ be a free Borel action and G_a its graph. If $f \colon X \to \mathbb{Z}$ is a Borel function and ϕ is a Borel f-flow for G, then there is an* integral *Borel f-flow ψ such that $|\phi - \psi| \leq 3^d$.*

Theorem (5.5 (Gao, Jackson, Krohne, Seward))**.** *Suppose $d \geq 1$. Let $a \colon \mathbb{Z}^d \curvearrowright X$ be a free Borel action on a standard Borel space X (i.e., $\mathbb{T}^k$). Let G_a be its graph and let $n \in \mathbb{N}$. Then there is a Borel set $C \subseteq X^{<\infty}$ such that $\bigcup C = X$, for every distinct $R, S \in C$ we have that ∂R and ∂S are at least at distance n in G_a and every $R \in C$ is connected in G_a.*

Theorem (6.1 (Gao, Jackson))**.** *Suppose $d \geq 1$. Let $a \colon \mathbb{Z}^d \curvearrowright X$ be a free Borel action on a standard Borel space X (i.e., $\mathbb{T}^k$). Let $n \in \mathbb{N}$. Then there is a Borel set $C_n \subseteq X^{<\infty}$ such that C_n partitions X, and every $S \in C_n$ is of the form $\{(n_1, \ldots, n_d) \cdot_a x : 0 \leq n_i < N_i\}$, where each N_i is either n or $n+1$ and $x \in X$.*